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Position-Space Renormalization for Systems with Weak Long-Range Interactions and the Breakdown of Hyperscaling

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Exact renormalization group equations are derived for a position-space renormalization of spin systems with weak long-range forces. It is shown how an apparent dependence of the critical exponents on the choice of the renormalization group can be resolved via the mechanism of "dangerous irrelevant variables" and that this same mechanism is responsible for the breakdown of hyperscaling. The dimension $d = 4$ can be seen to be a borderline dimension above which classical critical exponents are expected.

KEY WORDS: Renormalization; long-range interactions; scaling; dangerous irrelevant variables; redundant variables; hyperscaling; saddle point method; cumulant expansion.

1. INTRODUCTION

Hyperscaling may be expressed by scaling laws involving the dimensionality d of the system,[1] e.g.,

$$\alpha = 2 - d\nu \tag{1}$$

where α and ν are the critical indices characterizing the singular zero-field specific heat and correlation length behavior near the critical point. These scaling laws are not generally valid.[2] For example, the standard set of so-called *classical* critical indices, among them $\alpha = 0$ and $\nu = \frac{1}{2}$, does not satisfy (1) when $d \neq 4$.

While renormalization theory in its standard form (see, e.g., Ref. 3) seems to imply hyperscaling, at least one mechanism, which would allow

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hyperscaling to be violated, can be devised[4] within the theory. This is the presence of so-called "dangerous" irrelevant variables, which may rescale the singular free energy. Our motivation for the present study is that it would be of considerable interest to see more explicitly what happens in a model which on the one hand violates hyperscaling and on the other hand is amenable to a renormalization-group treatment without the usual truncation approximations. Spin systems with *weak long-range forces*[5] are such models. They can be analyzed completely in the limit of the strength and the inverse range of the interaction simultaneously going to zero, and they constitute in this limit mean-field models with classical indices.

We use position-space renormalization transformations[(6)] between the Ising spin Hamiltonian $H(s)$ and a Hamiltonian $H'(s')$ for collective cell spins s_i':

$$e^{H'(s')} = \sum_{\{s\}} T(s', s) e^{H(s)} \tag{2}$$

There is a considerable freedom both in the grouping of neighboring lattice sites into cells, whereby the length scale of the transformation is defined, and in the choice of the weight function $T(s', s)$ in the conversion of the site-spin problem into the cell-spin problem. We do not make a definite choice for the transformation; on the contrary, it will turn out to be crucial for the analysis that we keep this freedom.

In Section 2 we define the models under consideration and note in particular that, as far as thermodynamics is concerned, weak long-range forces in the infinite-range limit are equivalent to uniform interactions scaled down by the size of the system. Section 3 is devoted to a derivation of the exact renormalization recursion relations for the latter model by a saddle point method. The renormalization equations appear in a form from which the fixed point, which includes n-spin couplings for arbitrary, even n, can be directly determined. The renormalization equations are subsequently brought into a linear form by passing to new coordinates, which are the generalized scaling fields[(7)] for the problem. The conditions under which there exists a regular relationship between the old and the new coordinates are discussed.

The thermodynamic scaling behavior resulting from the so-introduced renormalization equations is discussed in Section 4. It is noted that the usual[(6)] relations between critical exponents and eigenvalues give rise to critical exponents that seem to depend on the choice of the renormalization transformation. However, we proceed to show that it is precisely the freedom in choice of the renormalization transformation that implies the singular behavior of certain scaling functions, i.e., the presence of "dangerous"

[4] Apparently first proposed by Fisher.[(4)]

[5] For references see the review by Hemmer and Lebowitz.[(5)]

irrelevant variables. Taking into account this singularity leads to new critical exponents corresponding to the correct classical behavior.

In order to obtain the scaling behavior of the correlation length one must discuss the long-range limit more carefully. In Section 5 we use a cumulant expansion which enables us to calculate the renormalization equations for long-range potentials that are not uniform, leading to an equation governing the transformation of the interaction range. In a discussion of the scaling behavior of the correlation length it is then found that it is this range that acts now as a dangerous variable and brings about a relation between the critical exponent ν and the eigenvalues of the renormalization group that differs from the usual one, thereby violating the hyperscaling relation (1).

In the last section we review our main results. As a second consequence of the transformation law for the range of the potential we find that the infinite-range fixed point is attractive for finite-range potentials when $d > 4$. This is a well-known result[8] of the ϵ-expansion, but which, to our knowledge, has not previously been found in the context of a position-space renormalization group.

2. MODEL

The spins of the model are Ising spins $s_{\mathbf{r}_j}$, localized on a regular d-dimensional lattice. The class of Hamiltonians we consider,

$$-\beta\mathscr{H}(s) \equiv H(s) = \sum_{n=1}^{\infty} \sum_{(\mathbf{r}_1,\ldots,\mathbf{r}_n)} J_n(\mathbf{r}_1,\ldots,\mathbf{r}_n) s_{\mathbf{r}_1} s_{\mathbf{r}_2} \cdots s_{\mathbf{r}_n} \tag{3}$$

contain translation-invariant weak interaction potentials of long range,[5]

$$J_n(\mathbf{r}_1,\ldots,\mathbf{r}_n) = \gamma^{d(n-1)} j_n(\gamma\mathbf{r}_{21}, \gamma\mathbf{r}_{31},\ldots, \gamma\mathbf{r}_{n1}) \tag{4}$$

where γ is a small parameter. The limit $\gamma \to 0$ is considered, taken in such a way that the range γ^{-1} remains small compared with the linear size of the system. The conditions on the functions j_n are that the interactions should be ferromagnetic in nature and should possess finite total strengths

$$J_n = \sum_{\mathbf{r}_2,\ldots,\mathbf{r}_n} J_n(\mathbf{r}_1,\ldots,\mathbf{r}_n) \tag{5}$$

and well-defined ranges

$$R_n = \left[\sum_{\mathbf{r}_2,\ldots,\mathbf{r}_n} (\mathbf{r}_{2,1} + \cdots + \mathbf{r}_{n,1})^2 J_n(\mathbf{r}_1,\ldots,\mathbf{r}_n)/J_n \right]^{1/2} \tag{6}$$

By (4), J_n does not depend upon γ, while $R_n \propto \gamma^{-1}$.

As far as calculations of thermodynamic quantities are concerned, the $\gamma \to 0$ limit can be shown by standard methods to be equivalent to using distance-independent potentials, scaled down with an appropriate power of the size of the system. Assuming each spin configuration to occur only once in the Hamiltonian (3), the equivalent uniform interaction Hamiltonian for a system of N spins is

$$H(s) = Ne(m), \tag{7}$$

with

$$m = N^{-1} \sum_{i=1}^{N} S_{\mathbf{r}_i} \tag{8}$$

and the energy function $e(m)$ is defined by

$$e(m) = \sum_{n \geqslant 1} J_n m^n / n! \tag{9}$$

In order to facilitate the comparison with what we will find later, it is useful to recall the standard solution for a system with such a uniform potential. If one introduces the entropy function

$$s(m) = -\frac{m+1}{2} \ln \frac{m+1}{2} - \frac{1-m}{2} \ln \frac{1-m}{2} \tag{10}$$

which is the entropy (per spin) of a free system with fixed magnetization m, then the free energy is given by

$$f = \max_{\{m\}} [e(m) + s(m)] \tag{11}$$

in the thermodynamic limit.

Expanding in powers of m, one obtains the usual Landau expression

$$f = \ln 2 + \max_{\{m\}} \left[\frac{1}{2} (J_2 - 1) m^2 + \frac{1}{4!} (J_4 - 2) m^4 + \cdots \right] \tag{12}$$

which has a classical critical point at $J_2 = 1$ provided that $J_4 < 2$. Above the critical point ($J_2 < 1$) the maximum is located at $m = 0$ and the free energy equals ln 2.

3. RENORMALIZATION OF THE UNIFORM INTERACTION HAMILTONIAN

For a uniform interaction Hamiltonian (7)

$$H(s) = Ne(m) \tag{13}$$

we first calculate the cell-spin Hamiltonian $H'(s')$ for a transformation (2), which dilutes the degrees of freedom with a factor

$$N'/N = l^{-d} \tag{14}$$

We assume the weight function $T(s'; s)$ to be a product of weight functions $t(s'; s_1, \ldots, s_{l^d})$, one for each cell:

$$T(s'; s) = \prod_{j'=1}^{N'} t(s'_{j'}; s_{j',1}, \ldots, s_{j',l^d}) \tag{15}$$

with a partition-function-conserving normalization

$$t(+; s_1, \ldots, s_{l^d}) + t(-; s_1, \ldots, s_{l^d}) = 1 \tag{16}$$

We first perform all summations over the site spins $s_i = \pm 1$ under the constraint that their sum equals Nm. By insertion of a Kronecker delta function

$$\delta_{\sum s_i, Nm} = (2\pi i)^{-1} \int_{-\pi i}^{+\pi i} dy \exp y\left(\sum_{i=1}^{N} s_i - mN\right) \tag{17}$$

these summations can be done independently:

$$\sum_{\{s\}} T(s'; s) e^{H(s)} = \sum_m e^{Ne(m)} \int_{-\pi i}^{\pi i} (dy/2\pi i) e^{-ymN} \prod_{j'=1}^{N'} Z(s'_{j'}) \tag{18}$$

The product extends over all cells, with, for cell j', the factor

$$\begin{aligned} Z(s'_{j'}) &= \sum_{s_1, \ldots, s_{l^d}} t(s'_{j'}; s_1, \ldots, s_{l^d}) \exp y(s_1 + \cdots + s_{l^d}) \\ &\equiv \exp[a(y) + b(y) s'_{j'}] \end{aligned} \tag{19}$$

The functions $a(y)$ and $b(y)$ are characteristics for the renormalization transformation chosen and $b(y)$ is expressed in terms of the weight function as

$$\exp 2b(y) = \frac{\sum_{s_1, \ldots, s_{l^d}} t(+; s_1, \ldots, s_{l^d}) \exp y(s_1 + \cdots + s_{l^d})}{\sum_{s_1, \ldots, s_{l^d}} t(-; s_1, \ldots, s_{l^d}) \exp y(s_1 + \cdots + s_{l^d})} \tag{20}$$

The function $b(y)$ may be considered as the renormalized magnetic field of a single (independent) cell in magnetic field y. The normalization condition (16) relates $a(y)$ and $b(y)$ to each other as

$$a(y) = l^d \ln(2 \cosh y) - \ln[2 \cosh b(y)] \tag{21}$$

In this section we will restrict ourselves to weight functions that are invariant under a simultaneous reversal of cell and site spins:

$$t(+; s_1, \ldots, s_{l^d}) = t(-; -s_1, \ldots, -s_{l^d}) \tag{22}$$

which implies that $b(y)$ is an odd function of y. Moreover, we assume that $t(\pm; s_1, \ldots, s_{l^d})$ is such that the derivative $\dot{b}(y)$ is bounded by

$$0 < \dot{b}(y) < l^d \tag{23}$$

implying that $b(y)$ is a monotonically increasing function of y.

Returning to the calculation of the renormalized Hamiltonian $H'(s')$, we see by inserting (19) into (18) that $H'(s')$ is a function of

$$m' = \sum_{j'=1}^{N'} s'_{j'}/N' \tag{24}$$

This allows us to write

$$H'(s') = Ng + N'e'(m') \tag{25}$$

where the constant g has to be chosen such that $e'(0) = 0$. From (18) we then find that

$$\exp[Ng + N'e'(m')] = \sum_m \int_{-\pi i}^{\pi i} (dy/2\pi i) \exp[Ne(m) - Nym + N'a(y) + N'b(y)m'] \tag{26}$$

For $N \to \infty$, the integral over y may be performed by the saddle-point method (the saddle point will be located on the real axis). Moreover, in this limit the sum over m reduces effectively to the maximum term:

$$e'(m') = \operatorname*{Max}_{m\in[-1,1]} \{l^d e(m) - l^d ym + l^d \ln(2\cosh y) - \ln[2\cosh b(y)] + b(y)m'\} \tag{27}$$

The saddle-point condition for y gives

$$l^d(\tanh y - m) + [m' - \tanh b(y)]\dot{b}(y) = 0 \tag{28}$$

while the maximum term condition for m requires that

$$y = \dot{e}(m). \tag{29}$$

The derivatives in (28) and (29) are denoted by a dot to avoid confusion with transformed quantities. Equations (28) and (29) may be used to express y and m in terms of m'. Inserting $y(m')$ and $m(m')$ into (27), one has found the renormalized energy function $e'(m')$ and the constant g. Thus (27) constitutes the renormalization transformation; at least this is the case provided that (28) and (29) allow a *unique* solution. We come back to this question at the end of this section, where it will be shown that there exists such a unique solution in a finite region around the fixed point.

In order to study the possible fixed points of (27), we transform the equation by considering $e(m)$, or rather its inverse μ

$$\mu(y) = \dot{e}^{-1}(y) \tag{30}$$

as representing the Hamiltonian. This choice is advantageous because we find for the renormalized derivative $\dot{e}'(m')$ from (22)

$$\dot{e}'(m') = b(y) \tag{31}$$

using that the implicit dependence of y and m on m' may be neglected in view of (28) and (29). Equation (31) is again a form of the renormalization transformation when y is expressed in terms of m'. By inverting (31) as

$$\mu'(b(y)) = m' \tag{32}$$

we obtain the renormalization transformation more transparently by now expressing m' in terms of y through use of (28) and the inverse of (29). Replacing $b(y)$ by x and therefore y by $b^{-1}(x)$, one finds for (32)

$$\mu'(x) = \tanh x + l^d\{\mu(b^{-1}(x)) - \tanh b^{-1}(x)\}/\dot{b}(b^{-1}(x)) \tag{33}$$

This form of the renormalization transformation enables us to see immediately that

$$\mu^*(x) = \tanh x \tag{34}$$

is a fixed point of the renormalization transformation. Converting to the $e(m)$ language, it means that

$$e^*(m) = \tfrac{1}{2}(1 + m)\ln(1 + m) + \tfrac{1}{2}(1 - m)\ln(1 - m) \tag{35}$$

is invariant under the renormalization transformation. The result (35) is quite remarkable, as it shows that the fixed point is completely *independent* of the choice of the renormalization transformation, i.e., independent of the arbitrary $b(y)$.

Moreover, one sees from the solution (11) for the free energy that $e^*(m)$ is related to the function $s(m)$ as

$$e^*(m) = -s(m) + \ln 2 \tag{36}$$

Thus at $e^*(m)$ *all* the coefficients in the Landau expression (12) vanish. This is in fact the source of the rather special nature of the scaling behavior of this model to be discussed in the next section.

The form (33) for the renormalization transformation is particularly suited for the discussion of the structure of this transformation around the fixed point. Writing

$$\mu(x) = \mu^*(x) + \psi(x) \tag{37}$$

one finds for the transform of $\psi(x)$

$$\psi'(x) = \mathscr{R}_b\psi(x) \tag{38}$$

where the *linear* operator $\mathscr{R}_b$ is defined as

$$\mathscr{R}_b\psi(x) = l^d\psi(b^{-1}(x))/\dot{b}(b^{-1}(x)) \tag{39}$$

The form (39) is still a complete representation of the original transformation given, e.g., by (27). The linearity of $\mathscr{R}$ allows us to discuss its properties in terms of its eigenfunctions and eigenvalues. The nth eigenfunction $\psi_n(x)$ can be characterized by

$$\psi_n(x) = x^{n-1} + O(x^{n+1}) \tag{40}$$

with associated eigenvalue

$$\lambda_n = l^d/\dot{b}(0)^n \tag{41}$$

One may construct the full eigenfunction recursively from its small-x behavior by

$$\psi_n(b(x)) = [\dot{b}(0)^n/\dot{b}(x)]\psi_n(x) \tag{42}$$

In fact, (41) is the second remarkable result because the eigenvalues depend on the choice of the renormalization transformation through $\dot{b}(0)$. The situation found here is therefore opposite to the situation one usually expects in renormalization theory, where the fixed point depends on the choice of the transformation while the eigenvalue structure is supposedly independent.

The action of the renormalization transformation on a general function $\mu(x)$ follows from the expansion

$$\mu(x) = \mu^*(x) + g_1\psi_1(x) + g_2\psi_2(x) + \cdots \tag{43}$$

where the fields g_n may be seen as Wegner's[7] scaling fields. Representing the Hamiltonian in terms of the scaling fields, one has

$$\mathscr{R}_b(g_1, g_2, \ldots) = (\lambda_1 g_1, \lambda_2 g_2, \ldots) \tag{44}$$

To make the transition back to the original Hamiltonian representation $e(m)$, we have to work out the connection between the Taylor expansion coefficients J_n and the scaling fields g_n. This can be achieved by a power series expansion and leads for the even coefficients to the formulas

$$J_2 = 1/(1 + g_2), \qquad J_4 = [2 - 6g_4 - \ddot{\psi}_2(0)g_2]/(1 + g_2)^4, \quad \ldots \tag{45}$$

Note that through $\ddot{\psi}_2(0)$ the choice of the transformation enters in this relationship. As the J_2 couples only to g_2, one derives easily the transformation for J_2 as

$$J_2' = J_2/[J_2 + \lambda_2(1 - J_2)] \tag{46}$$

The simplified representation (44) hinges only on the question of whether $e(m)$ is uniquely defined by a representation of the form (43) for $\mu(x)$. Since the construction of $e(m)$ from $\mu(x)$ involves an inversion, we need that $\dot{\mu}(x) > 0$. Assume now that the initial energy function is convex (which is the case if all couplings are ferromagnetic), so that $\dot{e}(m)$ may be inverted to $\mu(x)$. In terms of the functions μ the renormalization transformation is uniquely given by (33) or (43) and $\dot{e}'(m)$ can be constructed provided that

$\dot{\mu}'(x) > 0$. Since a unique construction of $e'(m)$ can only be expected when the saddle-point equations (28) and (29) have a unique solution, it is no surprise that the condition $\dot{\mu}'(x) > 0$ is at the same time a sufficient condition under which there is a unique saddle point. We see that one can safely use the representation (39) until one runs into a noninvertible $\mu(x)$. In terms of scaling fields, we have for inversion the condition

$$\dot{\mu}(x) = (\cosh x)^{-2} + g_2\dot{\psi}_2(x) + g_4\dot{\psi}_4(x) + \cdots > 0 \tag{47}$$

for the even subspace.

In order to analyze the condition (47) we have to know the behavior of the eigenfunctions $\psi_{2n}(x)$, which depends on the choice of the weight function through the form of $b(y)$. The properties that we need for a simplified discussion may be derived assuming

$$b(y) \simeq y + c, \qquad y \to \infty \tag{48}$$

and

$$b(y) > y \tag{49}$$

which are both conditions for which weight functions implying them can be easily found. The asymptotic behavior (48) leads upon insertion into (42) to the asymptotic form

$$\psi_{2n}(x) \simeq \exp\{2n[\ln \dot{b}(0)]x/c\} \tag{50}$$

implying that $\psi_{2n}(x)$ increases faster, the larger is n.

The recurrence relation (42) leads, furthermore, upon use of (49) to the conclusion that ψ_{2n} is increasing for all x provided that

$$\dot{b}(0)^{2n}/\dot{b}(x) > 1 \tag{51}$$

This last condition is fulfilled in the case that we will consider now, namely g_2 relevant ($\lambda_2 > 1$) and $g_4, g_6,\ldots$ not relevant ($\lambda_4, \lambda_6,\ldots \leqslant 1$), which implies $l^{d/2} > \dot{b}(0) > l^{d/4}$. In the other case ($\lambda_4 > 1$) one needs somewhat stronger conditions on $b(x)$ so that ψ_{2n} is increasing.

Returning now to the question under which conditions μ may be inverted, we see from (47) that this condition is fulfilled when all scaling fields are positive [since $\dot{\psi}_{2n}(x)$ is positive]. The case of interest will be that g_4, $g_6,\ldots > 0$ but that g_2 is allowed to be negative (but $g_2 > -1$), which corresponds to the subcritical region, as we shall see. For small values of x the first term in (47) will guarantee the positivity of $\dot{\mu}(x)$. For large values of x the g_4 term dominates the g_2 term in view of (50). The potential trouble may come from the intermediate x values. One needs a not too small positive value for g_4 to compensate for the negative g_2, i.e., one should demand

$$g_4 > M(g_2) \tag{52}$$

If one starts the renormalization procedure in the subcritical region where (52) is satisfied, one *cannot* repeat the renormalization transformation arbitrarily often without violating (52) since g_4 will decrease (as it is irrelevant) and $|g_2|$ will increase and one is bound to run into a noninvertible $\mu(x)$.

This fact is important if one tries to calculate the free energy. The free energy may be found recursively from the value of the constant g defined in (25) by use of the relation

$$f(e) = g + l^{-d}f(e') \tag{53}$$

which follows directly from the normalization condition (16). The value of the constant g may be found by setting $m' = 0$ in (28) and (29), which generally allows a solution $m_0 = y_0 = 0$. If this solution is *unique*, one obtains by insertion into (27)

$$g = (1 - l^{-d}) \ln 2 \tag{54}$$

When (54) applies to the whole renormalization trajectory one finds by iterating (53) for the free energy

$$f = \ln 2 \tag{55}$$

which agrees with the free energy found in the previous section in zero field above the critical point, but not with that found under subcritical conditions. The conclusion is that somewhere down the renormalization trajectory for $T < T_c$ the uniqueness condition is no longer met. However, as we shall see in the next section, one can still deduce the potential critical behavior from the renormalization transformations around the fixed point, where $\mu(x)$ may be inverted.

4. SCALING BEHAVIOR OF THE FREE ENERGY

As usual, the basis of the scaling behavior is the recurrence relation (53). In terms of the deviations ψ from the fixed point defined in (37), this relation takes the form

$$f(\psi) = l^{-d}f(\mathscr{R}_b\psi) + g \tag{56}$$

It is instructive to notice, following Wegner,[9] that two renormalization operators $\mathscr{R}$ and $\tilde{\mathscr{R}}$ corresponding to the same cell size l^d, but to different weight functions, are related by an "equivalence" operator $\mathscr{D}$ as

$$\mathscr{R} = \mathscr{D}\tilde{\mathscr{R}} \tag{57}$$

The "equivalence" operator $\mathscr{D}$ has the property that it leaves the free energy invariant

$$f(\psi) = f(\mathscr{D}\psi) \tag{58}$$

as may be seen by inserting (57) into (56).

In particular, we will use the operator $\mathscr{D}_b$, which connects the renormalization operator $\mathscr{R}_b$ with the renormalization operator $\mathscr{R}_1$ corresponding to a "decimation" weight function[10] defined by

$$t(s; s_1,..., s_n) = \delta_{s,s_1} \tag{59}$$

Such a decimation transformation corresponds, by (20), to $b(y) = y$, which upon insertion into (39) implies

$$\mathscr{R}_1\psi = l^d\psi \tag{60}$$

Consequently $\mathscr{R}_1$ is simply l^d times the unit operator. The equivalence operator

$$\mathscr{D}_b = \mathscr{R}_1^{-1}\cdot\mathscr{R}_b \tag{61}$$

takes the form

$$\mathscr{D}_b\psi(x) = \psi(b^{-1}(x))/\dot{b}(b^{-1}(x)) \tag{62}$$

and has the same eigenfunctions as $\mathscr{R}_b$:

$$\mathscr{D}_b\psi_n(x) = \dot{b}(0)^{-n}\psi_n(x) \tag{63}$$

Returning now to the scaling relation (56), we note that in terms of scaling fields it takes the form

$$f(g_2, g_4, g_6,...) = l^{-d}f(\lambda_2 g_2, \lambda_4 g_4, \lambda_6 g_6,...) \tag{64}$$

for the moment considering only the even subspace. Noting that

$$\lambda_n = l^d/\dot{b}(0)^n \equiv l^{d-\tau n} \tag{65}$$

it follows by the usual[11] arguments that the (singular part of) the free energy can be expressed in terms of a scaling function $\phi_\pm$ ($\pm$ referring to the sign of g_2) as

$$f_s(g_2, g_4, g_6,...) = |g_2|^{d/(d-2\tau)}\phi_\pm(g_4|g_2|^{(4\tau-d)/(d-2\tau)}, g_6|g_2|^{(6\tau-d)/(d-2\tau)},...) \tag{66}$$

This seems to imply critical exponents that depend on the choice of the renormalization group through the value of τ. This difficulty is resolved if one notices that the scaling function $\phi_\pm$ depends singularly on the scaling field g_4. In the case $\tau > d/4$, when g_4 is actually irrelevant this variable is therefore called a dangerous irrelevant variable.[4] The behavior of the function $\phi_\pm$ can be deduced from its scaling behavior under the action of the equivalence operator $\mathscr{D}_b$ defined above. Use of the invariance property (58) together with the transformation law (63) into Eq. (66) yields

$$\phi_\pm(g_4, g_6,...) = \mu\phi_\pm(\mu g_4, \mu^2 g_6,...) \tag{67}$$

with μ given by

$$\mu = l^{2\tau d/(d-2\tau)} \tag{68}$$

Consequently, ϕ can in its turn be written in terms of a scaling function $\tilde{\phi}$ as

$$\phi_{\pm}(g_4, g_6, \ldots) = (1/g_4)\tilde{\phi}_{\pm}(g_6/g_4^2, \ldots) \tag{69}$$

Inserting this relation into (66), we obtain for the free energy

$$f_s(g_2, g_4, g_6, \ldots) = (g_2^2/g_4)\tilde{\phi}_{\pm}(g_6|g_2|/g_4^2, \ldots) \tag{70}$$

From this relation it follows that the free energy in the limit $g_2 \to 0$, $g_4 > 0$ exhibits the usual mean-field-type singularity giving rise to the classical value $\alpha = 0$ for the specific heat critical exponent. At least this is the case provided that the scaling function $\tilde{\phi}_{\pm}$ does not diverge in this limit, i.e., $\tilde{\phi}_{\pm}(0, 0, \ldots)$ should be finite. From (70) it follows that

$$\tilde{\phi}_{\pm}(0, 0, \ldots) = (g_4/g_2^2) f_s(g_2, g_4, 0, 0, \ldots) \tag{71}$$

Recalling now the discussion at the end of the previous section, we see that, if one chooses $g_4 > M(g_2)$, the right-hand side of (71) is actually related to the free energy of a system with a properly defined energy function $e(m)$ and is therefore expected to be finite.[6] Notice that a similar reasoning for $\phi_{\pm}(0, 0, \ldots)$ fails because its value will be related to the free energy of a point $(g_2, 0, 0, \ldots)$.

It is possible to avoid the occurrence of a singular scaling function by a particular choice of the renormalization group operator. If one chooses $\tau = d/4$, the dangerous variable g_4 is left unchanged and the scaling relation (70) follows directly from (66). This choice of the renormalization group operators can be seen as the combined action of a renormalization and a properly chosen equivalence operator. A similar procedure was in fact already used by Wilson in his original renormalization group article,[(12)] where a renormalization step was combined with an appropriate spin rescaling step. The fixed point for this particular choice of the renormalization group operator is, for an arbitrary constant g_4^*,

$$\tilde{\mu}^* = \mu^* + g_4^* \psi_4 \tag{72}$$

with eigenvalues

$$\lambda_n = l^{d(1-n/4)} \tag{73}$$

The eigenvalue $\lambda_2 = l^{d/2}$ implies $\alpha = 0$, as already found from the scaling relations.

If we consider now also the space of *odd* interactions, we find a relevant eigenvalue $\lambda_1 = l^{3d/4}$ which, via the well-known relation

$$\delta = \ln \lambda_1/(d \ln l - \ln \lambda_1) \tag{74}$$

[6] This of course does not exclude the possibility that $\tilde{\phi}_{\pm}(0, 0, \ldots) = 0$, which amounts to a singularity with a zero amplitude, as is actually the case for $g_2 > 0$, i.e., for $T > T_c$.

implies the classical value $\delta = 3$. There is, however, a second relevant eigenvalue present in the odd space, namely $\lambda_3 = l^{d/4}$, corresponding to the scaling field g_3. This scaling field is, as we will show, a *redundant* scaling field, which means that new scaling fields $\tilde{g}_1, \tilde{g}_2, \ldots$ can be defined as analytic functions of the old fields so that

$$f(g_1, g_2, g_3, g_4, \ldots) = f(\tilde{g}_1, \tilde{g}_2, 0, \tilde{g}_4, \ldots) \tag{75}$$

In other words, g_3 can be transformed away (by an analytic transformation) and its eigenvalue does not bring about a new critical exponent. The presence of this variable is important, however, for the equation determining the critical surface because the latter is not given by $g_1 = g_2 = 0$, but by $\tilde{g}_1 = \tilde{g}_2 = 0$.

In order to show that g_3 is redundant, we use Wegner's original definition,[9] according to which redundant operators are operators obtained by applying the infinitesimal generators of the equivalence operators to the fixed-point Hamiltonian. Wegner shows that a renormalization group can be chosen such that the free energy does not depend upon the scaling fields of the so-defined redundant operators. From this the relation (75) follows directly.

In the present case the generator of the equivalence operator $\mathscr{D}_{I+f}$ applied to the fixed point $\tilde{\mu}^*$ defined in (72) yields, by expanding (62), a function ψ given by

$$\psi(x) = -g_4^*[f'(x)\psi_4(x) + f(x)\psi_4'(x)] \tag{76}$$

At this point we allow the weight functions used in defining $\mathscr{D}_{I+f}$ to be nonsymmetric, which does not affect the invariance property (58) since this is based entirely on the normalization of the weight functions. In that case $f(0) \neq 0$ and the Taylor expansion of $\psi(x)$ starts with a third-order term. The expression for the redundant operator[7] ψ in terms of the eigenfunctions ψ_n of $\mathscr{R}$ will therefore contain ψ_3. Since, as shown by Wegner,[9] the subspace of redundant operators is an invariant subspace under the action of $\mathscr{R}$, this implies that ψ_3 itself is a redundant operator. The redundancy of ψ_3 may be seen as a reflection of the well-known observation that a third-order term in the Landau expression (12) can be transformed away by a shift in m.

5. SCALING OF THE CORRELATION LENGTH

In order to discuss the behavior of the correlation length, it is no longer possible to consider only a uniform potential as we could in the case of the

[7] We stick to the commonly used expression "redundant operator," although in the present case ψ is a function.

thermodynamics. We shall in fact need to know how the range of a potential with a long, but finite, range transforms under the renormalization group. In addition, this will enable us to see when the "mean-field" fixed point discussed in the previous sections is attracting for potentials with finite ranges.

Consider first the Hamiltonian defined in (3) for two-body interactions only, i.e.,

$$H(s) = \sum_{\mathbf{r}_1, \mathbf{r}_2} J_2(\mathbf{r}_1 - \mathbf{r}_2) s_{\mathbf{r}_1} s_{\mathbf{r}_2} \tag{77}$$

with

$$J_2(\mathbf{r}) = \gamma^d j(\gamma \mathbf{r}) \tag{78}$$

We make use of the cumulant expansion[13] to see how (77) transforms under the renormalization group defined by

$$e^{H'(s')} = \sum_{\{s\}} \prod_{j'} t(s'_{j'};\, s_{j',1}, \ldots, s_{j',l^d}) e^{H(s)} \tag{79}$$

Expansion of the exponential in (79) transforms the problems into the calculation of cumulant diagrams with interaction bonds connecting cells j'. The basic observation is that all diagrams involving loops do not contribute to lowest order in γ. Thus the renormalized two-body interaction $J'(\mathbf{r})$ between two given cell spins is, to dominant order, given by the sum of all chain diagrams connecting these cells (see Fig. 1a). Since each term in this sum is actually a repeated convolution, it is convenient to describe the trans-

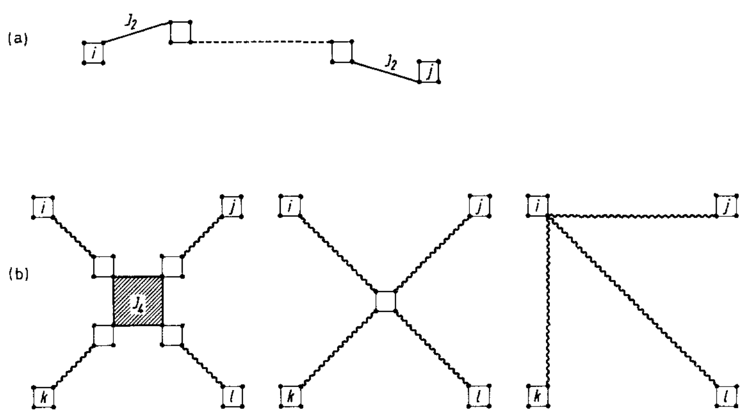

Fig. 1. Lowest order cumulant diagrams. (a) Example of chain diagrams contributing to $J_2'(\mathbf{r}_i, \mathbf{r}_j)$. (b) Typical diagrams contributing to $J_4'(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_l, \mathbf{r}_k)$. The wavy bond stands for any chain bond of the type depicted in (a).

formation in terms of the Fourier transform $\hat{J}_2(\mathbf{k})$ of $J_2(\mathbf{r})$. The transformed sum is a geometric series, and one finds finally

$$\hat{J}_2'(\mathbf{k}) = \frac{\hat{J}_2(\mathbf{k}/l)(\sum_i \langle s_i \rangle)^2}{l^d - \hat{J}_2(\mathbf{k}/l)[l^d - (\sum_i \langle s_i \rangle)^2]} \tag{80}$$

where $\langle s_i \rangle$ is the average of the ith site spin under the cell weighting function $t(s'; s_1, s_2, \ldots)$. Note that the sum in (80) is directly related to the function $b(y)$ introduced in (20) by

$$\sum_{i=1}^{l^d} \langle s_i \rangle = \dot{b}(0) \tag{81}$$

We can therefore rewrite (80) in terms of the eigenvalue λ_2, Eq. (41), as

$$\hat{J}_2'(\mathbf{k}) = \frac{\hat{J}_2(\mathbf{k}/l)}{\lambda_2 + (1 - \lambda_2)\hat{J}_2(\mathbf{k}/l)} \tag{82}$$

Setting $\mathbf{k} = 0$ in (82) yields the transformation of the integrated strength $\hat{J}_2(0) = J_2$ of the two-spin interaction, in agreement with the previously derived transformation (46) for uniform couplings.

In a similar way, by this cumulant method one can derive the renormalized four-spin interaction $J_4'(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$. The diagrams involved are indicated in Fig. 1b. We do not record the result here, but make the general remark that the integrated four-spin interaction strength also checks with that which follows from the relation (45) for the uniform potential model. The calculation of J_4' is tedious and it is not clear how to obtain the general J_n' (or even their fixed-point values J_n^*) by this cumulant method.

The transformation of the range R of the pair potential can be deduced from (82) via the relation

$$R^2 = \ddot{\hat{J}}_2(0)/\hat{J}_2(0) \tag{83}$$

from which it follows that at the fixed point $[\hat{J}_2(0) = 1]$ the transformed range R' is given by

$$R' = [l^{d/2-1}/\dot{b}(0)]R \tag{84}$$

Turning now to the behavior of the correlation length ξ, we write down the usual scaling relation[8,14]

$$\xi(H) = l\xi(H') \tag{85}$$

More explicitly, in terms of scaling fields,

$$\xi(g_2, g_4, g_6, \ldots; R) = l\xi(\lambda_2 g_2, \lambda_4 g_4, \lambda_6 g_6, \ldots; \lambda_R R) \tag{86}$$

where we introduced the range of the two-body forces as an extra scaling field with eigenvalue λ_R:

$$\lambda_R = l^{d/2-1}/\dot{b}(0) \equiv l^{d/2-(\tau+1)} \tag{87}$$

We restrict the discussion to this one extra scaling field representing the second moment of J_2, although in principle also the higher moments of J_2 and the moments of J_4, J_6, etc., could be taken into account. If one analyzes the scaling relation (86) in a similar way as that of the free energy in the previous section, one finds that ξ can be expressed in terms of a scaling function $X_\pm$ as

$$\xi(g_2, g_4, g_6, \ldots; R) = \left(\frac{g_2{}^2}{g_4}\right)^{-1/d} X_\pm(g_6|g_2|/g_4{}^2, \ldots; R g_4^{-1/d} g_2^{(4-d/2d)}) \qquad (88)$$

If $X_\pm$ would behave properly, this would imply a correlation length exponent ν given by $\nu = 2/d$, which satisfies the hyperscaling relation (1). However, in interpreting (88) one should realize that the cumulant approximation used to derive it becomes exact only in the limit $R \to \infty$. One expects that for a noncritical system the correlation length will diverge proportionally to R in that limit. This implies for the scaling function $X_\pm$ that for large R

$$X_\pm(g_6, \ldots; R) = \tilde{X}_\pm(g_6, \ldots) R \qquad (89)$$

where $\tilde{X}$ is now a well-behaved scaling function. Inserting this relation into (88) yields

$$\xi(g_2, g_4, g_6, \ldots; R) = R g_2^{-1/2} \tilde{X}_\pm(g_6|g_2|/g_4{}^2, \ldots) \qquad (90)$$

which implies the classical value $\nu = \frac{1}{2}$, violating hyperscaling when $d \neq 4$. We conclude that in the scaling relation for the correlation length it is the range R (or rather its inverse $1/R$) that plays the role of a dangerous variable. Just as in the case of the thermodynamics, it is also possible to choose a special renormalization group so that the usual relation

$$\nu = \ln l / \ln \lambda_2 \qquad (91)$$

is valid. In the present case the choice $\tau = d/2 - 1$ leaves the range unchanged and yields $\nu = \frac{1}{2}$ in (91).

6. CONCLUSION

We have seen that the breakdown of hyperscaling is connected with the presence of "dangerous" variables. The way in which these variables occur within this position-space renormalization theory is connected with the special position of the fixed point for which all coefficients in the corresponding Landau free energy vanish. One might say that this fixed point contains multicritical phenomena to arbitrary order. In this paper we restricted ourselves to ordinary critical behavior.

We have further demonstrated that the invariance operator $\mathscr{D}$ is important for the discussion of both "redundant" and "dangerous" operators. When the fixed point of a renormalization transformation is left invariant by an invariance operator $\mathscr{D}$, "dangerous" operators are to be expected and the singular structure of the scaling functions may be obtained from their transformation laws under $\mathscr{D}$. The other possibility (which is, one hopes, more common) is that the fixed point is *not* invariant under $\mathscr{D}$. The direction into which it moves is then a redundant direction.

In the present analysis dangerous variables occur generally both in the thermodynamics and in the correlation length. Among all possible renormalization transformations there are two special classes. The first class, with $\dot{b}(0) = l^{d/4}$, has the property that the scaling field g_4, which is a dangerous scaling field for the free energy, is not rescaled. The thermodynamic exponents follow in that case directly from the usual relations. However, for the correlation length the range R now acts as a dangerous variable and brings about a value for ν that differs from the usual expression that would have implied hyperscaling. The second class of renormalization transformations are those with $\dot{b}(0) = l^{d/2-1}$. The situation is then reversed; the correlation length exponent can be obtained directly but the derivation of the thermodynamic exponents needs to be refined due to the presence of the dangerous variable g_4.

We close this paper by noting that also in this position-space renormalization-group treatment $d = 4$ emerges as the dimensionality above which classical exponents are expected[(8)] for some class of finite-range potentials. Consider again Eq. (88) for a finite[8] R and note that in the limit where the critical point is approached (i.e., in the limit $g_2 \to 0$) this equation connects the correlation length for a system with finite R with that of a system of $R = \infty$ provided that $4 - d < 0$, i.e., $d > 4$. A similar statement can be seen to hold for the free energy by repeating the scaling analysis for the free energy of a system with finite R. Consequently, one expects the classical exponents obtained in this paper for infinite-range potentials to be valid also for finite-range potentials, with R sufficiently large, when $d > 4$. How large this R should be depends upon the possible existence of other fixed points for finite R beyond which Eq. (88) is no longer valid.

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[8] R should be large enough so that (88) is applicable when R is replaced by a generalized scaling field R.

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